KŪSHYĀR IBN LABBĀN GĪLĀNĪ'S TREATISE ON THE DISTANCES AND SIZES OF THE CELESTIAL BODIES

MOHAMMAD BAGHERI*, JAN P. HOGENDIJK** & MICHIO YANO ***

Kūshyār ibn Labbān Gīlānī (or al-Jīlī) was a well-known Iranian mathematician and astronomer who lived in the early eleventh century C.E. ([30], [31, V, pp. 343-345; VI, pp. 246-249], [28, pp. 414-420], [29, pp. 118-119], see the bibliography at the end of this paper). He composed his major astronomical work *al-Zīj al-Jāmiʿ* (The comprehensive astronomical handbook with tables) around 1020-1025 C.E. The work consists of four "books", which are entitled I: Elementary calculations, II: Tables, III: Cosmology, and IV: Proofs. The Arabic text of Books I and IV has been edited and published with English translation and commentary in [4], with a detailed account of Kūshyār's life and works and a comprehensive bibliography of modern research on him.

Kūshyār regards the first two books as the applied part of the *Zīj* and the next two books as its theoretical part. Book I consists of calculation methods used in astronomy and Book IV provides proofs for the validity of these methods. Book II contains the astronomical numerical data arranged in 55 tables. Book III consists of 30 chapters on different astronomical subjects. Chapter 22 of Book III is on the distances and sizes of the celestial bodies according to the methods used by Ptolemy (ca. A.D. 150) [34]. The thirty Chapters of Book III are followed by two extra unnumbered chapters. The first extra chapter, on the definition of astronomical terms, was also widely circulated as an independent work. It has been edited and translated in [3].

In the present paper, we translate and analyze the second (and last) extra chapter which Kūshyār appended to Book III of his *Zīj*. In this

* Institute for the History of Science, University of Tehran, Iran, email: mohammad.bagheri2006@gmail.com

** Mathematics Department, University of Utrecht, Netherlands, email: J.P.Hogendijk@uu.nl

*** Kyoto Sangyo University, Kyoto, Japan, email: yanom@cc.kyoto-su.ac.jp

second extra chapter, Kūshyār discusses how the sizes and distances of the celestial bodies can be determined. This chapter complements the earlier (short) Chapter 22 where only the results are stated without computation. The extra chapter can also be read and studied separately, as a more or less self-contained treatise on sizes and distances of the celestial bodies. Thus it was also copied and circulated as an independent work entitled *Risāla fi'l-abʿād wa'l-ajrām* (Treatise on the distances and (sizes of) the (celestial) bodies).

Kūshyār's treatise on the celestial distances and sizes is historically important for two reasons. First, the content is interesting. Although Kūshyār's text is to a large extent based on the *Almagest* and the *Planetary Hypotheses* of Ptolemy, Kūshyār deviates from Ptolemy in interesting ways. Most of Kūshyār's values for the sizes and distances are slightly different from those of Ptolemy, and Kūshyār presents what may be interpreted as a method for the determination of the length of the earth's shadow cone, using four suitably chosen lunar eclipses. This method makes it theoretically possible to determine the lunar distance from the earth without parallax measurements (as Ptolemy had made), although Kūshyār does not give details.

Secondly, Kūshyār's treatise belongs to a long tradition in medieval Islamic astronomy. Before him, al-Kindī (d. ca. 873 C.E.) discussed the determination of the distance of the moon from the earth [29, p. 40, no. A11]. Treatises on the celestial distances and sizes with the same title as that of Kūshyār had also been written by the following authors: Ḥabash al-Ḥāsib (d. ca. 870 C.E.) [31, V, p. 276], whose treatise is partly extant and has been published [19]; Abū Jaʿfar al-Khāzin (d. ca. 970 C.E.) [31, V, p. 299], whose treatise is lost but mentioned by al-Bīrūnī [8, p. 1312]; al-Qabīṣī (10th c. C.E.) whose treatise is extant but unpublished [31, V, p. 312]; and al-Ṣāghānī (d. 990 C.E.) [31, V, p. 311] whose treatise is also extant but unpublished (see also [29, pp. 28, 82, 85, 89]). In Kūshyār's time, al-Bīrūnī provided numerical information in Book III of his *Introduction to the Art of Astrology* [6, pp. 117-119], and a full account in Chapter 6 of Book X of his *Masʿudic Canon* [8, pp. 1301-1314]. Ibn Sīnā (d. 1037 C.E.) may also have written on the subject [29, p. 125, no. A12]. Quṭb al-Dīn Shīrāzī (d. 1310-11 C.E.) described Kūshyār's methods and parameters relating to the distances and sizes of the celestial bodies, and criticized Kūshyār's deviation from Ptolemy (see the commentary below). The tradition continued in the treatise *Sullam al-Samā*ʾ (Ladder of the Heaven) by al-Kāshī (d. 1436 C.E.), which is extant [29, p. 271, no. A4], and in a Persian text *Ab*ʿ*ād wa ajrām* ("Distances and Sizes") by al-Bīrjandī (d. 1525 C.E.) [29, p. 316, no. A14].

Kūshyār's treatise exists in nine Arabic manuscripts, two of which are written in Hebrew characters. We now list the manuscripts with the symbols we have used in the Arabic edition (compare [31, VI, p. 248], [29, pp. 118-119]).

F: Istanbul, Fatih, MS 3418/1, fols. 1v-175v [16, p. 472], Books I-IV, copied in 545 A.H./1150-51 C.E.; the treatise on distances and sizes is found on 125v-131r.

L: Leiden Universiteitsbibliotheek, MS Or. 8, fols. 1v-124r ([35, p. 405], [10, III, pp. 84-86, no. 1054], Books I-IV, copied in 634 A.H./ 1236-37 C.E.); the treatise on distances and sizes is found on fols. 99v-101v.

M: Moscow, Russian State Library, MS 154/3, fols. 36v-111r [21, II, p. 217], Books III and IV, copied in 525 A.H./1130-31 C.E.; the treatise on sizes and distances is found on fols. 69r-73r.

A: Alexandria, Baladiyya Library, MS 4285 jīm, fols. 1v-73v [37, pp. 216-217]), Books III and IV, copied in 566 A.H./1170-71 C.E. from an autograph dated 393 A.Y./415 A.H./1025 C.E.; the treatise on distances and sizes is found on fols. 27r-30r, but the first half of the treatise is missing in this manuscript.

N: Holon, MS Naḥum 209, pp. 1-64 [20, p. 151], Book III in Hebrew characters; the treatise is found on pp. 55-64.

K: Bankipore, Khuda Bakhsh Oriental Public Library in Patna (India), MS 2468/6 (now 2519), fols. 45r-47v, copied in 632 A.H./1234 C.E. [1, p. 64], [12, p. 139]; this ms. contains the treatise as an independent work.

B: Birmingham, Mingana collection, MS Arabic 1496 [13, p. 356, no. 1917], 48 fols. This manuscript contains excerpts from Kūshyār's *Zīj* and his astrological treatise, copied in the 19th century. An abridged version of Kūshyār's treatise on the distances and sizes is found on fols. 9v-14r of this manuscript; some parts are missing and in some other parts, only the numerical results are provided.

C: Cambridge University, MS Gg, 3/27 [9, pp. 93-94]; the treatise on the distances and sizes is found on fols. 52r-53v.

H: Jerusalem, National and University Library, MS 28° 6032 [20,

p. 151]; Kūshyār's treatise on the distances and sizes in Hebrew characters is found on fols. 60v-65v.

We note that the manuscript Tehran, Majlis library, MS 6451, mentioned in [31, VI, p. 248], does not in fact contain a copy of Kūshyār's *Zīj*.

An uncritical edition of **K** appeared in Hyderabad, India [17]. A Persian translation of this treatise was published by Bagheri in Iran [18]. In August 1993, he presented a paper on this treatise to the 19th International Congress of History of Science, Zaragoza. Yano made an edition of the manuscript **K** in which he corrected the errors in the Hyderabad publication. He added an edition of Chapter 22 of *al-Zīj al-Jāmi*ʿ (based on **F**, **K** and another manuscript of the *Zīj*, namely Istanbul, Yeni Cami, 784/3) with a Japanese translation of the treatise and the chapter, preceded by an introduction in Japanese [36]. This work has not been published. In the mean time, Bagheri also prepared a critical edition of the treatise, with draft translation and commentary. The two editions were compared and the translation and commentary were revised by Hogendijk. We are glad to present the result of all this research in this paper under joint authorship.

The Arabic edition in this paper is essentially the one by Bagheri. The manuscript **F** has been used as a basis for the edition. Significant variations in other Arabic manuscripts are provided in the apparatus, where "om" and "add" refer to words that were omitted or added in the indicated manuscripts. Whenever a variant in another manuscript is preferable, it has been used in the text, but the reading of **F** is indicated in the apparatus in such cases. For easy comprehension, the Arabic text has been divided into paragraphs, but no punctuation has been added. In the English translation we have made some additions in parentheses ( ). In the text, translation and commentary, the figures are not drawn to scale; i.e., they do not give an adequate proportional representation of the distances between the earth and the different celestial bodies as computed by Kūshyār.

Kūshyār's style is sometimes difficult to understand and his text is not free from inadequacies. Some of these errors are explained in the commentary, but we cannot claim to have resolved all difficulties. Throughout the treatise, Kūshyār seems to be interested in rough values for the sizes and distances of the celestial bodies, for he does not pay much attention to accuracy of his computations.

Numbers in the text are often indicated in words, as in "nineteen". We have not hesitated to use Hindu-Arabic numbers such as 19 in our translation. An expression such as "thirty-one degrees and fifteen minutes" has been rendered in the standard sexagesimal transcription as $31;15°$. In the commentary we have used a notation such as $10;19^r$ to indicate $10 + \frac{19}{60}$ times the unit denoted as ^{r} (i.e., earth-radii). For sake of clarity, we have sometimes rendered fractions such as 'one and one-fourth and one-fifth" in sexagesimal transcription as $1;27$.

Translation of Kūshyār's Treatise on the Distances and Sizes.

(This is) the treatise which we promised to produce and establish at the end of this Book. It is a commentary on Chapter 22 (of Book III of the *Zīj*) on the magnitudes of the distances and (sizes of) the (celestial) bodies according to Ptolemy's methods, and on the way to determine them.

I have seen that most people have often heard the astronomers say that a planet is in a certain sign and a certain degree, and that an eclipse (occurs) in such and such a (moment of) time. So they have become accustomed to these statements from them (i.e., the astronomers), and they have accepted the idea that there may be a way to (find) these (data). Now if it is said that the distance between the earth and one of the planets is such and such, and that the magnitude of its body is such and such, they shake their heads and lips and they really think that this is impossible (to find out). It seems to them that there is no way to these (determinations) unless by ascending towards them (i.e., the planets) and by getting close to their bodies and measuring them by their hands, as other objects on the earth can be measured. Among the group (of people having this false opinion) there are (even) some skilled in this art (of astronomy), whose opinion in this regard is close to their (i.e., the laypersons') opinion, because they have not progressed in the art to such an extent that they may regard this (determination) possible. And (even) if they regard it as possible, they (think) that it is difficult to derive something like it (i.e., the distances and sizes) and they make a big fuss of it. So I produced this treatise on the method (to find the magnitudes) of the (celestial) distances and bodies, the way to obtain them, and on what (in determining these distances and sizes) depends on observation and what is known by means of geometry and computation. God grants success and help.

Measurement of the earth.

Since the earth is in the middle of the heavens and the circularity of its surface is parallel to the circularity of the heavens, it happens that if one of us travels under one of the meridian circles towards the north or the south, the pole of the celestial equator will ascend or descend depending on the distance covered by the traveller. The distance on the surface of the earth corresponding to one degree was found (to be) $66\frac{2}{3}$ miles according to Ptolemy's methods. A mile is (equal to) 3,000 cubits; a cubit (is equal to) 36 digits; a digit (is equal to the width of) six barleycorns laid in a row with their bellies (adjacent) to each other. If the amount for one degree, which is $66\frac{2}{3}$ miles, is multiplied by 360, the circumference of the earth under one (meridian) circle is found (to be) 24,000 miles. Archimedes showed that the ratio of the diameter of any circle to its circumference is approximately equal to the ratio of 7 to 22, that is, 1 part of $3\frac{1}{7}$. If we multiply 24,000 by 7 and divide (the product) by 22, the result is the diameter of the earth, 7,636 miles, and its radius is 3,818 miles. The radius of the earth is used (as a unit) in measuring the other distances, and its body (i.e., volume) is used (as a unit) in measuring (the volumes of) the other bodies.

The distance of the moon from the earth.

The radius of (the moon's) epicycle, based on (assuming) its center at the apogee of the eccentric orb was found to be $5\frac{1}{4}$ parts, as is found by observation, and (the distance) between the center of the parecliptic orb (i.e., the center of the earth) and the center of the eccentric orb (was found to be) 10;19 parts, based on (taking) the radius of the parecliptic orb (equal to) 60 parts. The radius of the parecliptic orb was taken (as) the mean distance of the moon. Since the radius of the earth is one (part), its (i.e., the moon's) mean distance from the surface of the earth is 59 parts. If we add $5\frac{1}{4}$ parts to 60, then subtract one degree (i.e., one part) from it, the maximum distance of the moon from the surface of the earth is (found to be) $64\frac{1}{4}$ parts. If $5\frac{1}{4}$ parts is added to twice the (distance) between the two centers (of the earth and of the eccentric orb) which is (equal to) 20;38 parts, and the result is subtracted from 60 parts, the remainder is 34;7 parts. If one degree is subtracted from it (i.e., the remainder), its (i.e., the moon's) minimum distance from the earth is (found to be) 33;7 parts. It (i.e., moon's minimum distance) is the (upper) termination of (the realm of) the four elements (fire, air, water, earth) and the (lower) limit of the ether which is subject to the effects of the motions of the planets. Thus the maximum distance of the moon, which is used in what follows, and its minimum distance (which is not used in what follows) are known.

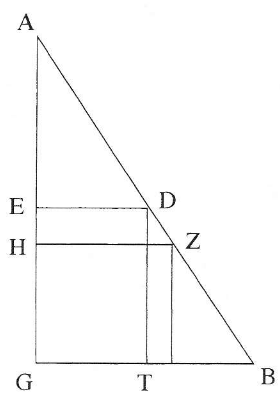
Which of the three bodies, i.e., the sun, the moon and the earth, is bigger than the others.

The sun is either smaller than the earth, or bigger than it, or equal to it. It is not smaller than the earth, because if it were smaller, then the earth's shadow would become thicker, up to infinity, as its altitude (i.e., distance) from the earth increases, and it would be thinnest at the earth. Then it would be necessary that the moon would be eclipsed at every opposition and would remain eclipsed the whole night (, which is not the case.) Nor is it (i.e., the sun) equal to it (i.e., the earth), because if it were equal to it, the shadow would have the same thickness as its altitude from the earth increases, and then it would be necessary for the moon what was necessary in the first (case), except that its duration (of total eclipse) would not be as much. Since it is not possible that the sun is smaller than the earth or equal to it, and (since), as the moon is higher (i.e., farther from the earth), the duration of a total (lunar) eclipse decreases, thus it is known that when the altitude (i.e., distance) of the shadow from the earth increases, it (i.e., the shadow) becomes thinner, and that the sun is therefore bigger than the earth.

The moon is smaller than the shadow (of the earth) where it passes through the shadow, because it remains (for some time completely) in the shadow (during a total lunar eclipse). But the shadow at that position is smaller than the earth. Thus the moon is much smaller than the earth.

The magnitude of the length of the (earth's) shadow and the magnitude of its diameter where the moon passes through it, and the magnitude of the diameter of its base (i.e., the diameter of the earth).

For this purpose, two lunar eclipses (occurring) near the ascending node (of the moon), at the maximum (lunar) distance were taken (into consideration). (The magnitude of) the first lunar eclipse was 3 digits based on (taking) the diameter of the moon 12 digits; its (i.e., the moon's) distance in longitude from the (ascending) node (was) $9\frac{1}{3}$ parts (i.e., degrees), and (its distance) in latitude (from the ecliptic was) 49 minutes and one-fifth. (The magnitude of) the second lunar eclipse was 6 digits; its distance in longitude from the (ascending) node (was) 7 parts and 48 minutes, and (its) (distance) in latitude (from the ecliptic was) 41 minutes and two-fifths. So the difference in digits (was) 3 digits, (the difference) in longitude (was) 1 part and 32 minutes, and (the difference) in latitude (was) 7 minutes and 48 seconds. Thus it became known that whenever the moon comes closer to the node by 1 part and 32 minutes in longitude and by 7 minutes and 48 seconds in latitude, the (magnitude of) the lunar eclipse increases by 3 digits. Then it becomes (clear that), if one regards the numbers rather than the degrees and minutes (see commentary below), the ratio of the difference in longitude (between the two eclipses) to the difference in latitude is equal to the ratio of the difference in digits to the complete obscuration (i.e., the radius of the shadow cone).



(Let) triangle ABG be half the triangle of the (section of the) shadow cone in longitude, AG the altitude of the shadow, DE the radius of the shadow at maximum distance of the moon, ZH its radius at the perigee of the epicycle, and BG the radius of the shadow's base (i.e., approximately the radius of the earth). BT (is equal to) the difference between DE and BG, DT is parallel to AG, and the line (segment)s DE, ZH, BG are parallel. So if we multiply the difference in digits by the difference in latitude, and divide (the product) by the difference in longitude, the complete obscuration (i.e. the radius of the shadow cone), which is DE, turns out to be approximately $15\frac{1}{2}$ digits.

(Now) similar to the two above mentioned lunar eclipses, if the two (eclipses) were in the same direction (with respect to the node) and (occurred) at the perigee of the epicycle, it became known that the radius of (the section of) the shadow there, which is the line (segment)

ZH, is $16\frac{1}{3}$ digits. Then it is known that for every $10\frac{1}{3}$ parts (i.e., earth radii), such as the epicyclic diameter EH - (if) the moon descends (this amount) from the maximum distance, the radius of the shadow (cone) increases by $\frac{1}{2}$ plus $\frac{1}{3}$ digits. If $64\frac{1}{4}$ is divided by $10\frac{1}{3}$ and the result is multiplied by $\frac{1}{2}$ plus $\frac{1}{3}$ digits, (the result BT) is approximately 5 digits. If it is added to $15\frac{1}{2}$, that is line (segment) DE, the line (segment) BG, the radius of the base of the shadow, is $20\frac{1}{2}$. The two triangles DTB and AGB are similar, and DT is equal to EG, so it is known. But TB and GB are (also) known. So AG, the altitude of the shadow (cone) is known. It is approximately 264 parts, if the radius of the earth is (assumed to be) one part.

The magnitude of the body of the moon in terms of the body of the earth.

It has been mentioned that the radius of the base of the shadow (of the earth) is $20\frac{1}{2}$ digits, and it is (equal to) the radius of the earth. If it is divided by the radius of the moon, which is 6 (digits), the result is 3 plus $\frac{1}{4}$ plus $\frac{1}{6}$. However, in ancient times they (i.e., the astronomers) computed its (magnitude) as $3\frac{2}{5}$. Thus the diameter of the earth is $3\frac{2}{5}$ times the diameter of the moon. It is demonstrated in the *Elements* (of Euclid, Proposition XII.18) that the ratio of (the volume of) a sphere to (the volume of another) sphere is equal to the ratio of the cube of the diameter (of the first sphere) to the cube of the diameter (of the other one). If $3\frac{2}{5}$ is multiplied for length, width and depth (i.e., cubed), the result is $39\frac{1}{4}$. So the body of the earth is $39\frac{1}{4}$ times the body of the moon (in volume).

The magnitude of the diameter of the sun at its mean distance in terms of the magnitude of the diameter of the moon at its maximum distance, and the distance of the sun from the earth.

When the disk of the moon was at its maximum distance (from the earth) and the disk of the sun was at its mean distance (from the earth), (they were) observed to be equal in very close approximation. Then, it was found by observation that the parallax of the moon at its maximum distance is $27\frac{1}{6}$ minutes, and the parallax of the sun at its mean distance is 1 plus $\frac{1}{4}$ plus $\frac{1}{5}$ minutes. If we interchange the positions of the two diameters,[1] so that we put (any) one of them in the other one's place,

[1]The manuscripts, except **A**, read: "the positions of the difference of the two diameters."

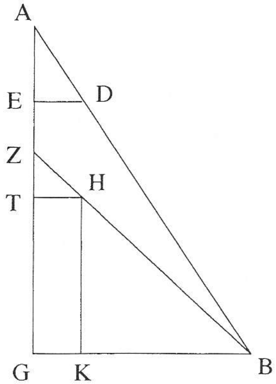
the ratio of the parallax (of the first body) to the parallax (of the second body) will be equal to the ratio of the diameter (of the second body) to the diameter (of the first body). Thus, if we divide 27 minutes and 10 seconds (= $27\frac{1}{6}$ minutes) by 1 minute and 27 seconds (=$1\frac{1}{4}+\frac{1}{5}$ minutes), the result is $18\frac{4}{5}$. Thus the diameter of the sun is $18\frac{4}{5}$ times that of the moon. By this (same inverse) ratio, the ratio of the diameter (of the first body) to the diameter (of the second body) is equal to the ratio of the distance (of the second body) to the distance (of the first body). So if we multiply the maximum distance of the moon, which is $64\frac{1}{4}$, by $18\frac{4}{5}$, the mean distance of the sun turns out to be approximately 1,208 parts, where the radius of the earth is one part. The distance between the two centers of the sun (i.e., the center of the eccentric orb and the center of the parecliptic orb, i.e., the center of the earth) is $2\frac{1}{2}$ (if the radius of the eccentic orb is 60), according to Ptolemy's methods. If we multiply it (i.e., $2\frac{1}{2}$) by $18\frac{4}{5}$, the product is approximately 47 parts (this result is exact). If we add it to 1,208 parts, the maximum distance of the sun (from the earth) is found as 1,255 parts. If we subtract it (i.e., 47 parts) from 1,208 parts, the remainder is the minimum distance of the sun (from the earth), that is approximately 1,161.

The magnitude of the body of the earth in terms of the body of the sun.

It has been mentioned that the diameter of the earth is $3\frac{2}{5}$ times the diameter of the moon. If we take the distance of the moon (to represent) its diameter, in order to facilitate the calculation for it and also for the following (bodies), the diameter of the earth, in terms of this (unit) magnitude, is 218. If the distance of the sun, which is approximately 1,208, also is (i.e., represents) its diameter, it is $5\frac{1}{2}$ times the diameter of the earth. If we multiply it for length, width and depth, the body of the sun is (found to be) 166 plus $\frac{1}{4}$ plus $\frac{1}{8}$ times the body of the earth.

The magnitude of the shadow of the moon.

Let triangle ABG be the triangle of the sun, BG the diameter of the sun, DE the diameter of the earth, and HT the diameter of the moon. We draw ZHB. Then TZ is the (length of the) shadow of the moon, and it is desired.



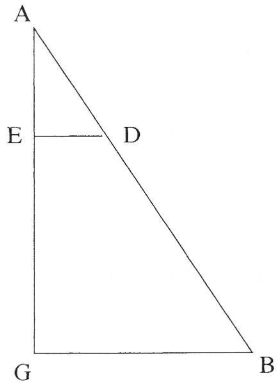
We draw HK parallel to ZG. Then the two triangles HBK and ZBG are similar. But GE is (equal to) 1,208 (parts) and TE (is equal to) $64\frac{1}{4}$ (parts). So TG (is equal to) 1,141 plus $\frac{1}{2}$ plus $\frac{1}{3}$, and it is equal to HK. So HK is known. But BG (is equal to) $18\frac{4}{5}$, and KG is (equal to) 1 (part) because it is equal to HT. So BK is $17\frac{4}{5}$. Thus ZG is known. But TG (is equal to) 1,141 plus $\frac{1}{2}$ plus $\frac{1}{3}$. So the remainder TZ is known, and it is found by calculation to be approximately equal to the maximum distance of the moon, that is $64\frac{1}{4}$ parts.

Mercury.

Its minimum distance from the earth was found equal to the maximum distance of the moon, because its parallax at minimum distance is equal to the moon's parallax at its maximum distance. The same was found for the situation of all the planets: the maximum distance of (any) lower (planet) is equal to the minimum distance of the higher (subsequent planet). (So) there is no need to repeat this statement for each one of them.

Then the (apparent) size of its (i.e., Mercury's) body was found: if it is one part at maximum distance, it was at minimum distance 2 plus $\frac{1}{3}$ plus $\frac{1}{4}$ (such parts). If we interchange the position of the sizes of the bodies, and we put (any) one of them in the position of the other, the ratio of the (first) body to the (second) body is equal to the ratio of the (second) distance to the (first) distance. So if we multiply the 2 plus $\frac{1}{3}$ plus $\frac{1}{4}$ by the maximum distance of the moon, and divide it by 1, (the result) is approximately 166 parts. This is the maximum distance of Mercury from the earth, where the radius of the earth is (assumed to be) 1 part. Then its (i.e., Mercury's) mean distance is 115, that is half the difference between the maximum and minimum distances added to the minimum distance.

Also, if the body of Mercury is compared with the body of the sun, while they are at their mean distances, it (i.e., the apparent diameter of Mercury) is one-fifteenth of the body (i.e., the apparent diameter) of the sun. Then we put the sun at the mean distance of Mercury and we see at what distance the body of Mercury becomes 1, in order that this distance will be (i.e., represent) its (i.e., Mercury's) diameter, as has been explained before in the cases of the moon, the earth and the sun.



(Let) in the triangle ABG point A (be) the earth and AG the mean distance of Mercury. BG is 15 and DE is 1 (part), and the line AE is desired. Then DE and BG are parallel, and the ratio of AE to ED is equal to the ratio of AG to GB. Each of (the line segments) AG, DE and BG is known, so AE is known, and it is (equal to) $7\frac{2}{3}$ parts. If the diameter of Mercury is $7\frac{2}{3}$ (parts) and the diameter of the earth is 218 (parts), the diameter of the earth is 28 plus a small amount times the diameter of Mercury. If we multiply it for length, width and depth, the body of the earth turns out to be 22,000 times the body of Mercury. This method can also be used for the remaining planets.

Venus.

(The ratio of) its (apparent) size at maximum and minimum distances is equal to 1 to 7 minus a small amount. So if 7 is multiplied by the maximum distance of Mercury, the product is 1,160, which is its (i.e., Venus') maximum distance and the sun's minimum distance. Its (i.e., Venus') mean distance is 663. Its body (i.e., its apparent diameter at mean distance) was compared with the body of the sun and found to be one-tenth of it. So if we divide 663 by 10, its diameter is found to be 66 plus $\frac{1}{5}$ plus $\frac{1}{10}$. If we divide this by the diameter of the earth (in fact, if we divide the latter, which is 218, by the former), (we find that) the diameter of the earth is $3\frac{1}{4}$ times that of it (i.e., Venus). If we multiply it for length, width and depth, the body of the earth turns out to be $34\frac{1}{3}$ times the body of Venus.

Mars.

(The ratio of) its (apparent) size at maximum and minimum distances is equal to 1 to 7, approximately like (that of) Venus. If we multiply the 7 by the maximum distance of the sun, its (i.e., Mars') maximum distance reaches 8,764. Its mean distance is 5,008. Its body (i.e., apparent diameter) was compared with the body of the sun while both were at their mean distances, and found to be one-twentieth of it (i.e., the apparent diameter of the sun). So if 5,008 is divided by 20, its diameter (i.e., the apparent diameter of Mars) is $250\frac{2}{5}$ parts. If we divide this by the diameter of the earth, the result is approximately 1 (part) plus 9 minutes. If this is multiplied for length, width and depth, the body of Mars turns out to be approximately $1\frac{1}{2}$ times the body of the earth.

Jupiter.

(The ratio of) its size at maximum and minimum distances is equal to 1 to 1 (part) and 37 minutes. If it is multiplied by the maximum distance of Mars, its (i.e., Jupiter's) maximum distance reaches 14,168. Its mean distance is 11,466. Its body (i.e., apparent diameter) was compared with the body of the sun while both were at their mean distances, and found to be one-twelfth of it (i.e., the apparent diameter of the sun). So if we divide its mean distance by 12, its diameter is $955\frac{1}{2}$. If we divide it by the diameter of the earth, its (i.e., Jupiter's) diameter is 4 plus $\frac{1}{4}$ plus $\frac{1}{6}$ times the diameter of the earth. If we multiply this for length, width and depth, the body of Jupiter turns out to be 84 plus $\frac{1}{4}$ plus $\frac{1}{8}$ times the body of the earth.

Saturn.

(The ratio of) its size at maximum and minimum distances is equal to 1 to $1\frac{2}{5}$. If it is multiplied by the maximum distance of Jupiter, its (i.e., Saturn's) maximum distance reaches 19,835. Its mean distance is

17,001. Its body (i.e., apparent diameter) was compared with the body of the sun while both were at their mean distances, and found to be one-eighteenth of the body (i.e., apparent diameter) of the sun. So if we divide its mean distance by 18, its (i.e., Saturn's) diameter is $944\frac{1}{2}$. If we divide it by the diameter of the earth, its diameter is $4\frac{1}{3}$ times the diameter of the earth. If we multiply it for length, width and depth, the body of Saturn turns out to be 81 plus $\frac{1}{5}$ plus $\frac{1}{6}$ times the body of the earth.

The fixed stars.

The distances of all of them are equal to Saturn's maximum distance, and their sizes have been observed in six magnitudes. For any (star) of the first magnitude, its body (i.e., apparent diameter) is one-twentieth of the sun's body (i.e., apparent diameter). If we divide its distance by 20, the diameter of each one of them is 991 plus $\frac{1}{2}$ plus $\frac{1}{4}$. If we divide it by the diameter of the earth, its (i.e., the star's) diameter is 4 plus $\frac{1}{2}$ plus $\frac{1}{2}$ of $\frac{1}{10}$ times the diameter of the earth. If we multiply it for length, width and depth, its body turns out to be $94\frac{1}{5}$ times the body of the earth. The stars below the first magnitude decrease (in size) gradually, so that finally at the sixth magnitude, their body is approximately 16 times the body of the earth. Thus the greatest bodies other than the (celestial) orbs are: the sun, then the fixed stars of the first magnitude, then Jupiter, then Saturn, then the remaining fixed stars according to their order (of magnitude), then Mars, then the earth, then Venus, then the moon, then Mercury.

The distances in miles.

The minimum distance of the moon, which is the termination of (the realm of) the four elements, is 126,440 miles.

The maximum distance of the moon, which is the minimum distance of Mercury, is 245,306 miles.

The length of the earth's shadow is 1,007,952 miles.

The maximum distance of Mercury, which is the minimum distance of Venus, is 633,788 miles.

The maximum distance of Venus, which is the minimum distance of the sun, is 4,428,880 miles.

The maximum distance of the sun, which is the minimum distance of Mars, is 4,783,954 miles.

The maximum distance of Mars, which is the minimum distance of Jupiter, is 33,460,952 miles.

The maximum distance of Jupiter, which is the minimum distance of Saturn, is 54,093,424 miles.

The maximum distance of Saturn, which is the distance of the fixed stars, is 75,730,030 miles.

These are the magnitudes of the distances and bodies, and the way to determine them.

After having fulfilled what we promised at the beginning of this book, we finish Book III by this treatise. Praise be to God, who is One and Sufficient, and His blessings be on Muḥammad, the Chosen one.

Commentary.

Measurement of the earth. Kūshyār's value of $66\frac{2}{3}$ miles for one degree on earth was attributed to Ptolemy by medieval Islamic geographers ([14, p. 131], cf. [27, p. 62]), and is unlike the values that were determined by medieval Islamic authors on the basis of their own observations. According to al-Bīrūnī, the astronomers of Caliph al-Ma'mūn found the terrestrial distance corresponding to one degree of geographical latitude equal to 56 or $56\frac{2}{3}$ miles [7, p. 179]. The noticeable difference between the magnitudes quoted by al-Bīrūnī and Kūshyār is due to the fact that a Roman and Syrian mile are shorter than an Arabic mile [23, p. 294]. For a detailed account of the subject see [23, pp. 412-416], [15, pp. 230-231].

The 24,000 miles for the circumference of the earth correspond to the 180,000 stades in Ptolemy's *Geography* VII, 5 [27, p. 110], and to approximately 38,340 km [23, p. 293], but of course the reader should bear in mind that the conversion into the metric system cannot be done very accurately because the lengths of the stade and the mile were defined with limited accuracy. The Ptolemaic magnitude for the circumference of the earth, as quoted by Kūshyār, leads to a value of approximately 12,200 km for the diameter of the earth. The difference from the modern value of the equatorial diameter of the earth is 4.3 %.

If the circumference of the earth is assumed to be equal to 24,000 miles, and $\pi = \frac{22}{7}$ is used, the diameter of the earth is $24,000/\pi \approx$ 7,636.363 miles, which Kūshyār rounds to 7,636 miles.

Distance of the moon from the earth. Kūshyār's lunar model and parameters are essentially the same as in Ptolemy's *Almagest* [26, pp. 220-234], [25, pp. 159-202]. The parecliptic orb is a circle in the plane of the ecliptic, carrying the lunar nodes. The center of this parecliptic orb coincides with the center of the earth. Kūshyār's eccentric orb is Ptolemy's moveable eccentric deferent [25, p. 186]. By the "mean distance" of the moon, Kūshyār means the distance between the center of the earth and the center of the lunar epicycle at the time of conjunction or opposition with the sun. At that time eclipses can take place, and the center of the lunar epicycle is at Kūshyār's parecliptic orb. Kūshyār first divides what he calls the "mean distance" into 60 parts, just as Ptolemy had done. We will use the notation ^{p} for these parts and ^{r} for earth radii. The values $5;15^p$ for the epicyclic radius and $10;19^p$ for the distance between the center of the earth and the center of the eccentric orb were derived by Ptolemy in *Almagest* IV.6 by a complicated mathematical argument on the basis of observations of lunar eclipses [26, pp. 190-203], [25, pp. 172-178], [24, pp. 73-79].

From Ptolemy's lunar theory it now follows that the maximum and minimum distance of the moon to the center of the earth are $60^p + 5;15^p = 65;15^p$ and $60^p - 5;15^p - 2 \times 10;19^p = 34;7^p$. These values can be easily deduced also from the figure for the lunar model provided by al-Battānī [5, III, pp. 77-80], whose work was well-known to Kūshyār.

Kūshyār then assumes that the radius of the earth is one 'part', without giving any reasons. In other words, he assumes $1^r = 1^p$. He then finds the maximum and minimum distance between the moon and the surface of the earth as $65;15^p - 1^r = 65;15^p - 1^p = 64;15^p$ and $34;7^p - 1^r = 34;7^p - 1^p = 33;7^p$. Further on in his text, he uses $64;15^r$ for the distance between the moon and the center of the earth. In Chapter 22 of his *Zīj*, Kūshyār gives the maximal and minimal lunar distance in earth-radii as $64;15^r$ and $33;7^r$. Thus his approach is confusing. Kūshyār explains in the following chapter how the distance of the moon in terms of earth-radii can be found without parallax observations.

In *Almagest* V.13 [26, pp. 247-251], [25, pp. 203-207], [24, pp. 100-103], Ptolemy argues, on the basis of an observation of lunar parallax, that $60^p = 59^r$, so $1^p = 0;59^r$. Ptolemy then finds the following maximal and minimal distances from the moon to the center (rather than the surface) of the earth: 59/60 times $65;15^p = 64;10^r$ and 59/60 times $34;7^p = 33;33^r$ [26, p. 251]. The maximum distance $64;10^r$ is also mentioned by al-Battānī [5, III, p. 90 line 2].

In Chapter 2, Book IV of his Persian treatise *Ikhtiyārāt-e Muẓaffarī* (MS 3074, National Library of Iran, copied in 1283-4 C.E., [2, pp. 728-729], see fol. 167v) Quṭb al-Dīn al-Shīrāzī criticizes Kūshyār for his deviation from Ptolemy's method and parameters. For Kūshyār's deviation from Ptolemy's lunar model, see also [4, p. 160].

The magnitude of the length of the earth's shadow and its diameter. The two lunar eclipses mentioned by Kūshyār are the same as those mentioned by Ptolemy in *Almagest* V. 14 [26, pp. 253-54] for the moon near the apogee of the epicycle. Note that they were observed by the Babylonians in the 6th and 7th century B.C.E., more than 1500 years before Kūshyār lived. Kūshyār incorrectly states that both eclipses occurred when the moon was near the ascending node, because the first eclipse occurred near the descending node. The two eclipses are mentioned in [25, pp. 207-209] and analyzed extensively in [24, pp. 104-108].

For the two eclipses, Ptolemy mentions the magnitude of the maximal obscuration of the moon in digits, where one digit is one-twelfth of the complete diameter of the moon. Since the magnitude of the second eclipse was six digits, the center of the moon at mid-eclipse was on the boundary of the shadow cone at mid-eclipse. This means that the radius of the shadow cone is equal to the distance of the center of the moon to the ecliptic, which distance must be measured perpendicularly to the lunar orbit (not perpendicularly to the ecliptic). Ptolemy approximates this distance by the latitude of the center of the moon (distance perpendicular to the ecliptic), which he computes from the longitude difference $7;48^\circ$ of the center of the moon at mid-eclipse from the node. He does not explain the details of the computation but only states that the resulting latitude is $0;40,40^\circ$. Similarly, he computes the latitude of the center of the moon at the moment of maximal obscuration of the first eclipse as $0;48,30^\circ$. The difference in maximal obscuration of 6 - 3 = 3 digits corresponds to one-quarter of the lunar disk, and Ptolemy assumes the difference to be equal to the latitude difference of $0;48,30^\circ - 0;40,40^\circ = 0;7,50^\circ$. He concludes that the apparent diameter of the lunar disk at maximum distance is $4 \times 0;7,50^\circ = 0;31,20^\circ$ and that the radius of the shadow cone (at a distance equal to the maximum lunar distance) is $0;40,40^\circ$, which is approximately $2\frac{3}{5}$ times the apparent radius $0;15,40^\circ$ of the lunar disk.

Kūshyār's discussion of the eclipse deviates from Ptolemy in different ways. First, his values of the latitude are different, and how he determined them is unclear to us. Using the standard method for finding the latitude of the moon from its argument, mentioned by al-Battānī [5, III, p. 113], the latitudes for the center of the moon at mid-eclipse for the two eclipses are $\beta_1 = 0;48,36°$ and $\beta_2 = 0;40,40°$. This method is equivalent to the modern formula $\sin\beta \approx \sin\eta \sin\beta_0$ with $\beta_0 = 5°$ the maximum latitude of the moon, and η the difference in ecliptical longitude between the moon and the node, with $\eta_1 = 9;20°$ and $\eta_2 = 7;48°$.

By linear interpolation in Kūshyār's own table of lunar latitudes (MS. Istanbul, Fatih 3418, fol. 80b) we obtain $\beta_1 = 0;48,40°$ and $\beta_2 = 0;41°$. Kūshyār knew that the substitution of latitude for distance (perpendicular to the lunar orbit) is an approximation and in I.4.9 and IV.4.8 in his *Zīj* [4, pp. 39-40, 151-52], he explains a more accurate method for finding the distance β' which is equivalent to the formula $\tan\beta' = \sin\eta \tan\beta_0$. Even if we apply this more accurate method, we find $\beta'_1 = 0;48,47°$ and $\beta'_2 = 0;40,49°$. Thus, Kūshyār's values $\beta_1 = 0;49,12°$ and $\beta_2 = 0;41,24°$ remain unexplained.

Secondly, Kūshyār determines the radius of the earth's shadow (at maximal lunar distance) R_d in digits, using an incorrect method. If we put d_1 and d_2 for the maximum obscuration in digits of the first and second lunar eclipses, his method boils down to the formula $(d_2 - d_1)/R_d = (\eta_1 - \eta_2)/(\beta_1 - \beta_2)$.

The following formula is correct (because all the arcs are small) $(d_2 - d_1)/R_d = (\eta_1 - \eta_2)/\eta_2 = (\beta_1 - \beta_2)/\beta_2$. Kūshyār nevertheless obtained a value of R_d which is close to the Ptolemaic value. This can be explained as follows.

Because $\eta_2 = 7;48°$ and $\beta_1 - \beta_2 = 0;7,48°$, we may state $\eta_2 = \beta_1 - \beta_2$ if we do not look at sexagesimal positions, but only at absolute numbers. Then $\eta_1 - \eta_2 = 1;32°$, and $1;32/7;48 = (9-6)/R_d$, so $R_d = 15.2608$ (which Kūshyār rounds to $15\frac{1}{2}$). Thus, the correct result of Kūshyār's computation can be explained by the correct formula $(d_2 - d_1)/R_d = (\eta_1 - \eta_2)/\eta_2$. Ptolemy's result, expressed in digits, is $R_d = 2\frac{3}{5} \cdot 6 = 15;36$ digits [26, p. 254], which differs only insignificantly from Kūshyār's value.

Kūshyār does not give details of the two lunar eclipses for the moon near minimum distance $54;15^p$ at conjunctions. In this situation the moon is near the perigee of the epicycle and the center of the epicycle near the apogee of the deferent, so this minimum should not be confused with the absolute minimum of approximately 33^p, which occurs near quadratures. Kūshyār probably used the two eclipses in *Almagest* VI.5 [26, pp. 283-285]. Based on the data given by Ptolemy, the radius of the shadow cone at mininum lunar distance $54;15^p$ can be deduced from these two eclipses by a method as above. The resulting radius of the shadow cone for the moon at minimum distance is $12 \times 0;46 : 0;35,20°$= 15.6226 digits, which is only negligibly greater than $2\frac{3}{5} \times 6$ digits. This would lead to the implausible consequence that the width of the shadow cone is the same for the moon at maximum and minimum distance. We do not know how Kūshyār arrived at his more realistic value of $16;20$ digits for the radius of the shadow cone at a lunar distance of $54;15^p$.

In the first figure presented in the text, Kūshyār uses the similarity of the triangles to find the length AG of the earth's shadow. Let K be the point of intersection of DT and ZH. By the similar triangles DBT and $DZK, TB = DT \times KZ/DK = 64\frac{1}{4} \times (16;20-15;30)/10;20 \approx$ $5.1814 \approx 5$. Thus $GB = GT + TB = ED + TB = 15\frac{1}{2} + 5 = 20\frac{1}{2}$. Here, Kūshyār uses the 'digit' as a unit of length (equal to one-twelfth of the lunar diameter) to measure the radius of the earth.

By the similar triangles DBT and ABG, $AG = GB \times DT/TB$. Apparently Kūshyār computed AG from the maximum lunar distance $DT = 64\frac{1}{4}^r$, as $(20\frac{1}{2} \times 64\frac{1}{4}^r)/5 = 264.425 \approx 264^r$. In *Almagest* V.15, Ptolemy [26, p. 257] determines the length of the shadow cone as 268^r by a different method.

Note that the fact that the distances AG and DT are in earth-radii is not important; they could equally well have been expressed in the 'parts' such that the radius of the parecliptic is 60^p as above. All we need to know is the ratio AG/DT, and thus the computation could be made without knowledge of the lunar parallax. Since $\angle DGE = 0;15,40°$ is also known from the eclipse observations above, it is now possible to compute the lunar distance EG to the earth in digits, and hence the ratio EG/GB, that is, the lunar distance in earth-radii. Thus, Kūshyār's method could be used, at least in principle, for computing the lunar distance on the basis of four suitably chosen lunar eclipse observations. Of course, the (in)accuracy of the resulting lunar distance would depend on the (in)accuracy of the eclipse observations.

Volume of the moon. Since the radius of the earth is $20\frac{1}{2}$ digits, the diameter of the earth is $41/12 = 3\frac{5}{12} = 3\frac{1}{4} + \frac{1}{6}$ times the diameter of the moon. Kūshyār prefers to use the Ptolemaic value $3\frac{2}{5}$ [26, p. 257], which value differs by only 0.5 % from his own result. The modern value for the ratio of the mean radius of the earth to the radius of the moon is $(6,378.388 + 6,356.912)/3,476 \approx 3.66$, which is slightly more than Ptolemy's and Kūshyār's values.

Kūshyār follows Ptolemy [26, p. 257] in giving the volume of the earth as $39\frac{1}{4}$ times that of the moon (a more accurate ratio of the volumes is 39.304 with Ptolemy's value and 39.885 with Kūshyār's value). The ratio of the volumes based on modern values is about 49.189.

The distance of the sun from the earth. Kūshyār's value for the ratio of the mean distance of the sun to the maximum distance of the moon ($18\frac{4}{5}$) is a Ptolemaic parameter. In *Almagest* V.15-16 [26, pp. 255-257], Ptolemy found this value by a famous geometrical argument involving lunar and solar eclipses, but Kūshyār simply derives the ratio from the ratio between the lunar parallax (which could be observed) and a solar parallax which he states to be $1 + \frac{1}{4} + \frac{1}{5}$ minutes = $0;1,27°$. Such a small angle could never have been observed in ancient and medieval times, so Kūshyār's approach is misleading. Ptolemy did not observe the solar parallax but computed it from the solar distance which he had obtained in *Almagest* V.15-16. We note that the Ptolemaic value for the solar distance is only 1/20 of the value according to modern measurements, and that the maximum parallax of the sun is slightly less than $0;0,9°$.

Kūshyār's above-mentioned values of the lunar and solar parallax are close to the Ptolemaic tabular values for the total lunar and solar parallax if the distance from the zenith is 30° [26, p. 265], namely $0;27,9°$ and $0;1,25°$, respectively. Kūshyār's value in his solar parallax table (table no. 51 in Book III of *al-Zīj al-Jāmiʿ*) is $0;1,29°$. Kūshyār computed the mean distance of the sun as $64\frac{1}{4}^r \times 18\frac{4}{5} = 1,207.9^r$, which he rounded to $1,208^r$. The Ptolemaic value is $1,210^r$ [26, p. 257]. The Ptolemaic values for the minimum and maximum distances of the sun are $1,160^r$ and $1,260^r$ [25, p. 394], [11, p. 7]. Kūshyār's values are $1,161^r$ and $1,255^r$. Later in our text, Kūshyār also uses $1,160^r$ and $1,252^r$ or $1,253^r$ for the minimal and maximal solar distance.

The fact that Ptolemy's and Kūshyār's values for the distance of the sun are approximately 20 times less than the true value, is only one of the reasons why the results of their calculations of the planetary distances are very different from the modern values. Another reason is the assumption (made by Ptolemy and Kūshyār) that the planetary spheres are adjacent in a geocentric model of the universe. We have therefore refrained from comparing Kūshyār's values for the sizes and distances of the planets with modern values.

The magnitude of the body of the earth in terms of the body of the sun. Kūshyār assumes that the maximal distance of the moon to the earth is $64;15^r$, and that the mean distance of the sun to the earth is 1210^r. Since the apparent diameters of the sun at its mean distance and of the moon at its maximum distance are equal, the ratio of the lengths of their diameters is equal to the ratio of their distances from the earth. This is true irrespective of the unit of length in which the lunar and solar diameters are expressed. Kūshyār now introduces a new unit of length, which we will call 1^u, in such a way that the diameter of the moon is $64;15^u$. This is what Kūshyār means when he says "we take the distance of the moon as its diameter". Because $64;15^u : 1210^u = 64;15^r : 1210^r$, the diameter of the sun, expressed in the new unit is 1210^u. In general, if a celestial body has an apparent diameter equal to that of the moon at maximum distance, and its distance to the earth is Δ^r for some number Δ, then the length of its diameter is Δ^u. This principle will be used below for the determination of the volumes of the planets. Kūshyār's method is an elaboration of a method mentioned by Ptolemy in the *Planetary Hypotheses* [11, p. 8, col. 2].

The diameter of the earth is $64\frac{1}{4}^u \times 3\frac{2}{5} = 218.45 \approx 218^u$ units; the factor $3\frac{2}{5}$ is the above-mentioned ratio between the diameters of the earth and the moon. Therefore the ratio of the diameter of the sun to that of the earth is $1,208/218 = 5.541 \approx 5.5$. Thus the ratio of their volumes can be found as $(5.5)^3 = 166.375 = 166\frac{1}{4} + \frac{1}{8}$. The Ptolemaic values for the ratios of the diameters and volumes of the sun and the earth are $5\frac{1}{2}$ and 170, respectively [26, p. 257].

The length of the shadow of the moon. This paragraph presents several problems which we have not been able to solve. They may have been caused by imperfect transmission of Kūshyār's text. Kūshyār actually computes the minimum length of the shadow of the moon, because he takes the moon at maximum distance $64;15^r$ from the earth. In the corresponding figure, the calculation of TG is incorrect because

$TG = GE - TE$ but $1208 - 64\frac{1}{4} \neq 1141 + \frac{1}{2} + \frac{1}{3}$. His own parameters would produce $TG = 1,143\frac{3}{4}^r$, and using Ptolemaic values one finds $TG = 1,210^r - 64\frac{1}{6}^r = 1,145 + \frac{1}{2} + \frac{1}{3}^r$. By similar triangles, $ZG = HK \times GB/BK = (1,141\frac{5}{6}^r \times 18\frac{4}{5})/17\frac{4}{5} = 1,205.981^r \approx 1,206^r$ and $TZ = 1,206^r - 1,141\frac{5}{6}^r = 64\frac{1}{6}^r$, that is the maximum lunar distance according to Ptolemy [26, p. 257]. Kūshyār does not provide computations but states the result as $64\frac{1}{4}^r$, that is to say, equal to his own maximum distance of the moon.

The equality of the minimum length of the shadow of the moon and the maximum distance of the moon from the earth implies that the apparent solar disc can be completely eclipsed by the lunar disc even in solar eclipses that occur at maximum distance of the moon. The same assumption was made by Ptolemy in the *Almagest*. The assumption is incorrect because annular solar eclipses are possible and were even observed in antiquity, see [24, p. 104].

The distance and volume of Mercury. Kūshyār adopts Ptolemy's onion-like model of the spheres of the celestial bodies for the minimum and maximum distances of the moon, Mercury, Venus, the sun, Mars, Jupiter and Saturn [11, pp. 4, 7, 9-11, 29-30]. This model is presented in Ptolemy's *Planetary Hypotheses*, which has been preserved in an Arabic translation (*Kitāb al-manshūrāt* or *Kitāb al-iqtiṣāṣ*) and in a Hebrew translation as well [11, p. 3]. In this model, the maximum distance of each celestial body is equal to the minimum distance of the next body in the above-mentioned order. For Ptolemy, this equality between maximum and minimum distance of successive planets is the result of a philosophical principle to the effect that there cannot be useless space. Kūshyār deduces the equality from alleged parallax observations, which show that the parallax of a planet at maximum distance is equal to the parallax of the next planet at minimum distance. Such parallaxes were impossible to observe for the sun and the planets. According to modern astronomical data, the planet which comes closest to the earth is Venus, at a minimum distance of approximately 40 million km. Thus the maximum parallax of any planet is less than $0;1^\circ$.

Kūshyār correctly states the mathematical principle that the ratio of the maximum distance of a planet to its minimum distance is (very closely) equal to the inverse of the ratio of its apparent diameters at these distances. We note that the maximum apparent diameters of

Venus and Jupiter are $0;1,6°$ and $0;0,50°$ according to modern astronomical data, so the apparent diameter of any of the five planets Mercury, Venus, Mars, Jupiter and Saturn as well as the fixed stars could not have been observed in ancient Greek and medieval Islamic astronomy. So again Kūshyār's account is misleading. Ptolemy derived the ratio between maximum and minimum distance from the geometrical model (eccenter with epicycle) of the planet. Kūshyār's values for the ratio between maximum and minimum distance are close to those of Ptolemy.

Just like Ptolemy in the *Planetary Hypotheses*, Kūshyār assumed that the apparent diameter of Mercury is a nice fraction, namely $1/15$, of the apparent diameter of the sun. In the geometrical figure, Kūshyār then attempts to determine the diameter of Mercury in the new unit of length ^{u} which he defined above for the computation of the volume of the sun. In the corresponding figure, BG is the diameter of a body at the mean distance $AG = 115^r$ of Mercury but with an apparent diameter equal to that of the moon at maximal distance and the sun at mean distance. Therefore $BG = 115^u$. Therefore the true diameter of Mercury is $115/15 = 7\frac{2}{3}^u$. Because the diameter of the earth is 218^u, the ratio of the diameter of the earth to that of Mercury is $218 : 7\frac{2}{3} \approx 28.43 \approx 28$. Thus the ratio of the volume of the earth to that of Mercury is $28^3 = 21,952 \approx 22,000$. For the Ptolemaic value see the table at the end of the next section of the commentary.

Note that the geometrical figure is confusing because DE does not represent a celestial body at distance AE. It would have been clearer to delete the line ED and take point D on line BG such that $GD : GB = 1 : 15$; then GD would represent the actual diameter of Mercury, which has to be found.

Mercury was a very small body in the Ptolemaic system because it is always seen as a point although it is close to the earth: its minimal distance was supposed equal to the maximum distance of the moon.

The distances and volumes of Venus, Mars, Jupiter, Saturn.

We write d for the minimum distance between any planet and the earth and D for its maximum distance. For each of the four planets Venus, Mars, Jupiter, and Saturn, Kūshyār uses as his fundamental data the ratio $d : D$ between minimal and maximal distance of the planet, and the ratio A between apparent diameter of the planet and the apparent diameter of the sun when the planet is at its mean distance

$\frac{1}{2}(d+D)$. A is always supposed to be a nice fraction. For Mercury, $d : D = 1 : 2\frac{1}{3}$ and $A = \frac{1}{15}$. Now d for any planet is D for the preceding planet, and the volume v can be computed as $(\frac{\frac{1}{2}A(d+D)}{218})^3$, as illustrated above in the case of Mercury. Here d and D are expressed in earth-radii and v is the the volume of the planet divided by the volume of the earth.

We now present the data and the results of Kūshyār's computations in tabular form.

Name	$d : D$	A	d	D	v
Moon	-	1	33;7	64;15	$1/39\frac{1}{4}$
Mercury	$1{:}2\frac{1}{3}+\frac{1}{4}$	1/15	64;15	166	1/22,000
Venus	1:7	1/10	166	1,160	$1/34\frac{1}{3}$
Sun	23:25	1	1,161	1,255	$166\frac{1}{4}+\frac{1}{8}$
Mars	1:7	1/20	1,255	8,764	$1\frac{1}{2}$
Jupiter	60:97	1/12	8,764	14,168	$84\frac{1}{4}+\frac{1}{8}$
Saturn	5:7	1/18	14,168	19,835	$81\frac{1}{5}+\frac{1}{6}$

We note that the maximum distance of Venus coincides with the minimum distance of the sun which had already been computed previously. This is consistent with Kūshyār's account of the planetary spheres nested inside one another.

Ptolemy had found the maximum distance of Venus $1{,}079^r$ and the minimum solar distance $1{,}160^r$. In order to overcome the contradiction with the philosophical principle about the impossibility of useless space, he discusses the possibility of decreasing the distance of the sun in such a way that its minimum value becomes equal to the maximum distance of Venus [11, pp. 4, 7]. Kūshyār avoided such difficulties.

Kūshyār's computations involve a certain amount of rounding. We illustrate his rounding procedure by a few examples in which we use decimal fractions, which were not used by Kūshyār. Sometimes the rounding is very good, as in his determination of the ratio of the diameter of the earth to that of Venus: $(218 \times 10)/663(\approx 3.288\ldots) \approx 3\frac{1}{4}$, and in the subsequent determination of the ratio of the volumes of Venus and the earth as $(3\frac{1}{4})^3(\approx 34.328\ldots) \approx 34\frac{1}{3}$.

In the case of Jupiter, the diameter of the body is $955.5/218 = 4.383\ldots$ times the diameter of the earth. In the text, this number is given as $4 + \frac{1}{4} + \frac{1}{6} = 4.41667\ldots$. On the basis of this value, the ratio between the volumes of Jupiter and the earth is $(4 + \frac{1}{4} + \frac{1}{6})^3 = 86.1557\ldots$ but Kūshyār provides the result $84 + \frac{1}{4} + \frac{1}{8}$. This value is the cube of $4.3860\ldots$, which number is very close to $955.5/218$. We have no explanation for Kūshyār's accurate volume determination.

In the case of Mars, note that $D = 1,255 \times 7 = 8,785^r$. Kūshyār's maximum distance $8,764^r$ corresponds to a maximum solar distance of $1,252^r$.

Kūshyār's results may be compared to those of Ptolemy in the *Planetary Hypotheses*, which we have listed in the following table:

Name	$d : D$	A	d	D	v
Moon	-	$1\frac{1}{3}$	33	64	1/40
Mercury	34:88	1/15	64	166	1/19,683
Venus	16:104	1/10	166	1,079	1/44
Sun	23:25	1	1,160	1,260	$166\frac{1}{3}$
Mars	1:7	1/20	1,260	8,820	$1\frac{1}{2}$
Jupiter	23:37	1/12	8,820	14,189	$82\frac{1}{2} + \frac{1}{4} + \frac{1}{10}$
Saturn	5:7	1/18	14,189	19,865	$79\frac{1}{2}$

On these values and the related problems one may consult [11, p. 9-12] and [22, p. 36-37].

The distances and volumes of the fixed stars. Just like Ptolemy, Kūshyār assumes that the distance between the fixed stars and the earth is the maximal distance of Saturn to the earth, and that the apparent diameter of a star of the first magnitude is equal to 1/20th of the apparent diameter of the sun. Kūshyār computes the volume of a star of the first magnitude as $94\frac{1}{5}$ times the volume of the earth. Ptolemy's value is $94\frac{1}{6} + \frac{1}{8}$ [11, p. 9].

Kūshyār states that the volume of stars of the sixth magnitude is 16 times that of the earth. This corresponds to a diameter of about 2.51 times that of the earth. The ratio of their apparent diameter to that of the sun must have been supposed equal to $2.51 \times 218 : 19,835 = 1/36.11\ldots \approx \frac{1}{36}$.

The classification of the visible stars into six magnitudes according to their brightness was introduced by Hipparchus (2nd c. B.C.) and adopted by Ptolemy [24, pp. 277-292] and hence also by the medieval Islamic astronomers. In modern astronomy, the system has become more precise and more extended.

Distances in miles. We reconstruct Kūshyār's computations below.
Minimum distance of the moon: $33\frac{7}{60} \times 3,818 = 126,439.433\ldots \approx 126,440$.
Maximum distance of the moon: $64\frac{1}{4} \times 3,818 = 245,306.5 \approx 245,306$.[2]
The length of the earth's shadow: $264 \times 3,818 = 1,007,952$.
Maximum distance of Mercury: $166 \times 3,818 = 633,788$.
Maximum distance of Venus: $1,160 \times 3,818 = 4,428,880$.
Maximum distance of the sun: $1,253 \times 3,818 = 4,783,954$.
We note that Kūshyār's value for the maximum distance of the sun is $1,253^r$ (instead of $1,255^r$ mentioned above).
Maximum distance of Mars: $8,764 \times 3,818 = 33,460,952$.
Maximum distance of Jupiter: $14,168 \times 3,818 = 54,093,424$.
Maximum distance of Saturn: $19,835 \times 3,818 = 75,730,030$.

In a similar vein, Ptolemy provides the celestial sizes and distances in stades [11, pp. 7-8], where one mile corresponds to 7.5 stades.

Acknowledgement. We are grateful to Dr. Hamid-Reza Giahi Yazdi for his comments on a preliminary version of this article, and to all friends and librarians who kindly made available manuscript copies from all over the world.

References

[1] Abdul Hamid, M., *Catalogue of the Arabic and Persian Manuscripts in the Oriental Public Library at Bankipore*, vol. 22 (Arabic mss.), Science, Patna, 1937.

[2] Anvār, A., *Fihrist-e nosakh-e khaṭṭī-e ketābkhāne-ye mellī* (A Catalogue of the Manuscripts in the National Library [of Iran]), Tehran, 1358 A.H.S./1979 C.E.

[3] Bagheri, M., Kūshyār ibn Labbān's Glossary of Astronomy, *SCIAMVS* **7** (2006), pp. 145-174.

[4] Bagheri, M., *Az-Zīj al-Jāmiʿ by Kūshyār ibn Labbān: Books I and IV*, Frankfurt: Institute for the History of Arabic-Islamic Science, 2009 = [32], vol. 114.

[2] In Chapter 22 of his *Zīj*, Kūshyār presents the maximum lunar distance as 245,255 miles, which value is probably the result of a computational error.

[5] Al-Battānī, *Opus astronomicum* (Kitāb al-Zīj al-Ṣābī), edition, Latin translation and commentary by Carolo Alphonso Nallino, 3 vols., Milano, 1899-1907; reprinted in [32], vols. 11-13, 1998.

[6] Al-Bīrūnī, *Introduction to the Art of Astrology*, ed. and tr. Ramsay Wright, London 1934; reprinted in [32], vol. 29, 1998.

[7] Al-Bīrūnī, *The Determination of the Coordinates of Positions for the Correction of Distances Between Cities: a translation from the Arabic of ... Taḥdīd Nihāyāt ... al-amākin*, tr. Jamil Ali, Beirut: American University of Beirut, 1967; reprinted in [33], vol. 26, 1992.

[8] Al-Bīrūnī, *Kitāb al-Qānūn al-Masʿūdī (Masʿudic Canon)*, Hyderabad-Deccan, Osmania Oriental Publications Bureau, 1956, 3 vols.

[9] Brown, E.G., *A Handlist of the Muḥammadan Manuscripts, Including All Those Written in the Arabic Character, Preserved in the Library of the University of Cambridge*, Cambridge, 1900.

[10] De Jong, P. & M.J. De Goeje, *Catalogus Codicum Orientalium Bibliothecae Academiae Lugduno-Batavae*, vol. III, Leiden, 1865-1866.

[11] Goldstein, B.R., The Arabic Version of Ptolemy's Planetary Hypotheses, *Transactions of the American Philosophical Society, New series,* **57** (1967), Part 4, pp. 3-55.

[12] Hogendijk, J. P., Rearranging the Arabic Mathematical and Astronomical Manuscript Bankipore 2468, *Journal for the History of Arabic Science* **6** (1982), pp. 133-59.

[13] Hopwood, D., *Catalogue of the Mingana Collection of Manuscripts*, vol. IV, Birmingham, 1963.

[14] Kennedy, E.S., *A Commentary upon Bīrūnī's Kitāb taḥdīd al-amākin*, Beirut, 1973; reprinted in [33], vol. 27, 1992.

[15] King, D. A., Too Many Cooks ... : A New Account of the Earliest Muslim Geodetic Measurements, *Suhayl* **1** (2000), pp. 207-31.

[16] Krause, M., Stanbuler Handschriften islamischer Mathematiker, *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, B **3** (1936), 437-532; reprinted in [32, 83, pp. 237-332], 1998.

[17] Kūshyār ibn Labbān, *Risālat fi'l-abʿād wa'l-ajrām*, in *Rasā'ilu'l-mutafarriqa fi'l-hai'at li'l-mutaqaddimín wa mu'ásiray al-Bírúní*, Hyderabad-Deccan, Osmania Oriental Publication Bureau, 1948; reprinted in [32], vol. 74, 1998.

[18] Kūshyār ibn Labbān, Resāle-ye abʿād va ajrām, Persian tr. Mohammad Bagheri, in Hezareh Gooshiar Gili, *Proceedings of the millennium of Kūshyār Gīlānī held in Gilan University*, Rasht, 1988, pp. 107-126.

[19] Langermann, Y.T., The Book of Bodies and Distances of Ḥabash al-Ḥāsib, *Centaurus* **28** (1985), pp. 108-128.

[20] Langermann, Y.T., Arabic Writings in Hebrew Manuscripts: A Preliminary Relisting, *Arabic science and philosophy* **6** (1996), pp. 137-60.

[21] Matvievskaya, G.P., & B.A. Rosenfeld, *Matematiki i astronomy Musulmanskogo srednevekovya i ikh trudy (VIII-XVIII vv.)* (Muslim mathematicians and astronomers of the Middle Ages and their works), Moscow: Nauka, 1983.

[22] Murschel, A., The Structure and Function of Ptolemy's Physical Hypotheses of Planetary Motion, *Journal for the History of Astronomy* 26 (1995), 33-61.

[23] Nallino, C. A., *Raccolta di scritti (editi e inediti), a cura di Maria Nallino,* vol. V, Roma, 1944.

[24] Neugebauer, O., *A History of Ancient Mathematical Astronomy*, New York: Springer, 1975, 3 vols., paginated serially.

[25] Pedersen, O., *A Survey of the Almagest,* Odense: Odense University Press, 1974.

[26] Ptolemy, *Ptolemy's Almagest*, translated by G. J. Toomer, Princeton: Princeton University Press, 1998, second edition.

[27] Ptolemy, *Geography, Annotated Tanslation of the Theoretical Chapters by J. L. Berggren & A. Jones*, Princeton: Princeton University Press, 2000.

[28] Qorbānī, Abo'l-Qāsem, *Zendegīnāme-ye riyāẑīdānān-e dowre-ye eslāmī*, Tehran: Markaz-e Nashr-e Dāneshgāhī, second edition, 1375 A.H.S./1996 C.E.

[29] Rosenfeld, B.A., & E. Ihsanoğlu, *Mathematicians, Astronomers & Other Scholars of Islamic Civilization and Their Works (7th-19th c.)*, Istanbul: IRCICA, 2003.

[30] Saidan, A.S., Article: Kūshyār ibn Labbān, in C.G. Gillispie, ed., *Dictionary of Scientific Biography*, vol. 7, pp. 531-533, New York: Scribner's Sons, 1973.

[31] Sezgin, F., *Geschichte des arabischen Schrifttums*, Band V: Mathematik bis ca. 430 H, Leiden: Brill, 1974; Band VI, Astronomie bis ca. 430 H, Leiden: Brill, 1978.

[32] Sezgin, F., ed., *Islamic Mathematics and Astronomy*, Frankfurt: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1997 - ..., 120 volumes to date.

[33] Sezgin, F., ed., *Islamic Geography*, Frankfurt: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1992 - ..., 318 volumes to date.

[34] Toomer, G.J., article: Ptolemy, in C.G. Gillispie, ed., *Dictionary of Scientific Biography*, vol. 11, pp. 186-206, New York: Scribner's Sons, 1975.

[35] Voorhoeve, P., *Handlist of Arabic Manuscripts in the Library of the University of Leiden and Other Collections in the Netherlands,* vol. VII, Leiden: Brill, 1957.

[36] Yano, M., Reseach Report, Project Number 14580008, Grant-in-Aid for Scientific Research, Japan, Kyoto Sangyo University, 2005, cf. http://kaken.nii.ac.jp/en/p/14580008/2004/6/en

[37] Zaydān, Y., *Fihrist makhṭūṭāt al-Maktaba al-Baladiyya fī l-Iskandariyya*, 6 vols., Alexandria, 1926-1929.